N-Dimensional Distributed Network Localization With Noisy Range Measurements and Arbitrary Anchor Placement

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Abstract— **This work presents a distributed algorithm for node localization problems in static sensor networks in n-dimensions. We focus on networks in which $n + 1$ nodes with known locations (anchors) are arbitrarily placed among all other nodes with unknown locations. In the noiseless case, barycentric coordinates computed from range measurements are used to transform the non-convex node localization problem into a standard linear system of equations. Meanwhile, adding independent zero mean Gaussian noise to range measurements turns all barycentric coordinates to dependent random variables with no known standard distribution which may not even be identically distributed. Relying on online optimization methods, we provide a distributed online gradient descent algorithm to solve the noisy range-only localization problem. Finally, comparisons among simple barycentric coordinate averaging, a centralized gradient descent formulation and our distributed algorithm are provided.**

I. Introduction

Deployments of mobile robotic sensors have tremendous potential to revolutionize information gathering and exploration in applications such as environmental monitoring, search and rescue, and security and surveillance. It is always the case that the sensors need to know their own locations with respect to a common frame of reference before they can utilize each other's measurements successfully and collaborate on higher level tasks. Relative localization is especially challenging but also most important to solve robustly in harsh operational conditions, where GPS access is denied, the surroundings are featureless, and the sensed information is lower-dimensional than the sensor states of interest (e.g., bearing-only or range-only measurements used for 3-D position and orientation estimation) and perturbed by significant measurement noise. To uniquely determine a frame of reference for the sensor network positions, it is usually required that the locations of a small subset of the nodes, called anchor nodes, are known [1]. The problem has been considered in planar, 3-D, or even higher-dimensional settings, with different types of measurements and noise distributions. Some methodologies utilize range and bearing measurements simultaneously [2], [3], while others rely on

bearing-only [4]–[7] or range-only [8]–[12] measurements. Another usual algorithmic classification relates to how unknown node locations are computed. If all available network information is sent to one computational node, it is called centralized [8]; else, if each node processes its own information along with data obtained from its communication neighbors and computes all node locations, it is called distributed [5], [13], [14].

Our goal is to provide a distributed algorithm capable of estimating node locations from noisy range-only measurements and arbitrary anchor configurations in n dimensions. Trilateration methods, such as [9] and [15], are perhaps the most common techniques for range-only localization problems. They solve a set of non-linear equations, in which measured distances between nodes must equal the Euclidean norm of their Cartesian coordinates. Thomas *et al.* [9] proposed a different set of equations based on barycentric coordinates instead of the standard Trilateration set to solve a two-dimensional localization problem. The advantage is that all non-linearities occur in the barycentric coordinate computation, while the final system of equations is *linear*. Barycentric coordinates are a set of $n + 1$ scalars which sum to one and uniquely determine the Cartesian coordinates of a point in a n-dimensional affine space in relation to a set of $n + 1$ points, which are not all contained in any sub-spaces of $n - 1$-dimensions. Their algorithm utilizes Cayley-Menger bi-determinants and determinants [16] in order to compute necessary coordinates based on a geometrical view of the 2-D problem. Trilateration methods have a weakness in that unlocalized nodes can be localized only if they are connected to a minimum of $n + 1$ previously localized nodes. Therefore, the sequence in which the nodes are localized is important. The following methods have lesser requirements on the node network.

Khan *et al.* [10], [17] provided a framework for solving localization problems in n-dimensional spaces whenever all unknown nodes are strictly inside the convex-hull of the anchor nodes and each unknown node also resides inside the convex-hull of $n + 1$ of its neighbors. These two assumptions allow them to compute barycentric coordinates from the absolute values of Cayley-Menger determinants. Diao *et al.* [11] relaxed the constraint on anchor and unknown node locations. Unfortunately, the later method is only defined for the 2D case and utilizes heuristics to determine the signed barycentric coordinates. Han *et al.* [18] proposed a hybrid method based on Multidimensional Scaling (MDS) and the framework proposed by Khan *et al.* [10], an approach for non-convex non-linear optimization over a set of trilateration

equations [12], [19], in order to overcome the difficulties of correctly obtaining signed barycentric coordinates for arbitrarily placed nodes in 3-D space. MDS is employed locally at each node to compute its barycentric coordinates with respect to its neighbors and global consistency is ensured using the approach of Khan *et al.* [10].

In this paper, we present a new formulation for computing barycentric coordinates in n-dimensions with arbitrary placement of anchor and unknown nodes. We make use of Cayley-Menger bi-determinant and determinant properties and relations to algebraic and geometric views of the problem. As a result, we deduce a formula for computing barycentric coordinates for any possible arrangement of a node and a configuration of its neighbors in n-dimensional space based only on Cayley-Menger bi-determinants and inter-node range measurements. Therefore, we can extend the approach of Diao *et al.* [11] to n-dimensional space. Even though current applications seem to be restricted to 3-dimensions, we can not discard possible applications in higher dimensional graphs in computer graphics and physics. The formula, however, is accurate only with noiseless range measurements, as it is computed via determinants of matrices consisting of range measurements. In particular, noisy measurements introduce a sequence of matrices that are neither independent nor identically distributed, and as a result, standard central limit theorems do not provide a solution in this scenario.

While in the centralized case online gradient descent [20] is applicable, to solve the problem in a distributed fashion, we use the distributed online mirror descent technique proposed in [21]. Due to the geometry of the localization problem, a suitable method turns out to be a distributed online gradient descent algorithm, which enables the nodes to collectively calculate barycentric coordinates from noisy range measurements. In summary, this paper makes the following contributions.

- We provide a closed-form expression for computing barycentric coordinates in n dimensions using Cayley-Menger bi-determinants.
- We develop an n-dimensional range-only localization algorithm that relies on $n+1$ *arbitrarily* placed anchor nodes.
- We develop a distributed algorithm capable of handling noisy range measurements with arbitrary noise distribution.

II. PROBLEM DEFINITION

A static sensor network in n-dimensional Euclidean space can be modeled as a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, with vertex set $\mathcal{V} = \{1, \ldots, m\}$. We refer to the Cartesian coordinates $x_i \in \mathbb{R}^n$ of node $i \in \mathcal{V}$ as its location. Let the set of node locations $\mathcal{X} := \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ be divided into a set $\mathcal{X}_a$ of anchor nodes with known locations and a set $\mathcal{X}_u$ of nodes with unknown locations such that $\mathcal{X}_a \bigcup \mathcal{X}_u = \mathcal{X}$ and $\mathcal{X}_a \bigcap \mathcal{X}_u = \emptyset$. An edge $(i, j) \in \mathcal{E}$, where $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, exists whenever nodes i and j are able to communicate and measure their relative distances, $d(\mathbf{x}_i, \mathbf{x}_j) = ||\mathbf{x}_i - \mathbf{x}_j||_2$. Supposing that each node $i \in \mathcal{V}$ has a maximum sensing range $r_i \in \mathbb{R}_{>0}$, an edge between node pair (i, j) exists if $d(\mathbf{x}_i, \mathbf{x}_j) < r_i$.
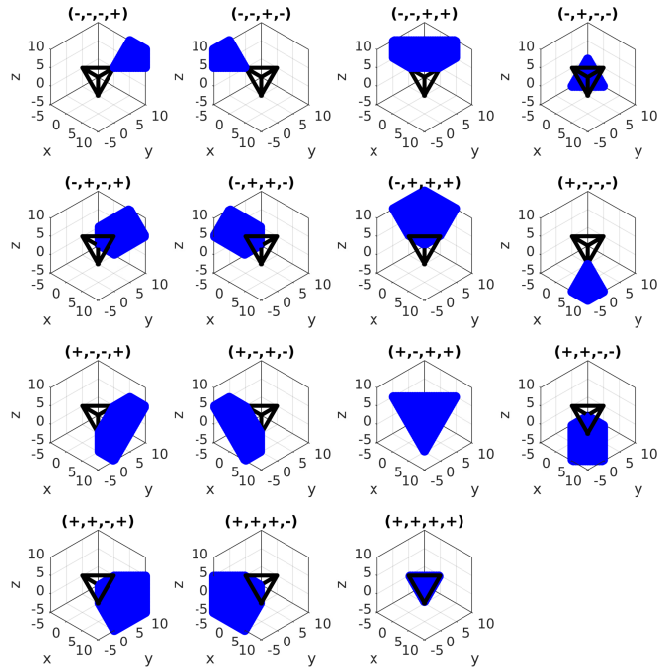

Fig. 1. Regions (blue) containing a node's location determined by the sign of its (non-zero) barycentric coordinates.

We suppose that each pair of nodes $(i, j) \in \mathcal{E}$ obtains range measurements, $z_{ij}(t) = d(\mathbf{x}_i, \mathbf{x}_j) + \delta_{ij}(t)$, over time $t = 1, \ldots, T$ perturbed by independent identically distributed (iid) measurement noise $\delta_{ij}(t)$ with an unknown probability distribution. Assuming $\delta_{ij}(t)$ and $\delta_{ji}(t)$ are independent random variables, the measured quantities $z_{ij}(t)$ and $z_{ji}(t)$ are distinct random variables and thus the generated graph is directed. Conversely, the noiseless case leads to an undirected graph formulation.

Problem. Given the anchor node locations $\mathcal{X}_a \subset \mathcal{X}$ of a static sensor network and noisy range measurements $z_{ij}(t) \in \mathbb{R}$, $\forall (i, j) \in \mathcal{E}$ and $t = 1, \ldots, T$, estimate the locations of all unknown nodes $\mathcal{X}_u \subset \mathcal{X}$.

The following sections present a centralized solution for the noiseless case based on barycentric coordinates (Sec. III), an approach for handling noisy measurements based on barycentric coordinate averaging (Sec. IV-A), a centralized gradient descent algorithm (Sec. IV-B) and, finally, a distributed gradient descent algorithm (Sec. V).

III. CENTRALIZED NOISELESS SOLUTION

In this section, we develop a centralized noiseless solution for the localization problem based on barycentric coordinates. We show how to use Cayley-Menger bi-determinants to compute generalized barycentric coordinates [11] and construct a linear system, whose solution determines all unknown node locations.

Definition 1 ([22, Prop.3.6.2 (modified)]). Let $\{\mathbf{x}_i\}_{i=0}^n$ be a

frame[1] for an affine space $\mathbf{X}$. For any point $\mathbf{x} \in \mathbf{X}$ there exist $\lambda_i \in \mathbb{R}$, $0 \le i \le n$, such that $\sum_i \lambda_i = 1$ and $\mathbf{x} = \sum_i \lambda_i \mathbf{x}_i$. The scalars λ_i are uniquely defined by this property and are called the barycentric coordinates of $\mathbf{x}$ in the frame $\{\mathbf{x}_i\}_{i=0}^n$.

The importance of Definition 1 is that it allows us to determine the location of an unknown node i using its barycentric coordinates with respect to $n+1$ neighbor nodes with known coordinates. Also, note that the signs of the barycentric coordinates of a node reveal a lot of information about its location in relation to its neighbors. If all signs are positive, then the node is located inside the convex hull of its neighboring nodes; if at least one coordinate is negative, then it is outside of the convex hull. A node's barycentric coordinates can also be zero, which would indicate that it lies on a hyperplane formed by the neighbors. In Fig. 1, we show the region of 3-D space that contains a node's location, determined based on the sign of its barycentric coordinates.

The barycentric node coordinates will play a fundamental role in the localization algorithms we develop and hence we need an efficient approach for computing them in arbitrary n-dimensional node configurations. For this purpose, we first introduce the notions of Cayley-Menger bi-determinant and determinant.

Definition 2. The Cayley-Menger bi-determinant of points $\{\mathbf{x}_i\}_{i=0}^n$ and $\{\mathbf{y}_i\}_{i=0}^n$ is defined as:

$$D(\mathbf{x}_0, \dots, \mathbf{x}_n; \mathbf{y}_0, \dots, \mathbf{y}_n) =$$
$$2\left(-\frac{1}{2}\right)^{n+1} \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & d(\mathbf{x}_0,\mathbf{y}_0)^2 & d(\mathbf{x}_0,\mathbf{y}_1)^2 & \dots & d(\mathbf{x}_0,\mathbf{y}_n)^2 \\ 1 & d(\mathbf{x}_1,\mathbf{y}_0)^2 & d(\mathbf{x}_1,\mathbf{y}_1)^2 & \dots & d(\mathbf{x}_1,\mathbf{y}_n)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d(\mathbf{x}_n,\mathbf{y}_0)^2 & d(\mathbf{x}_n,\mathbf{y}_1)^2 & \dots & d(\mathbf{x}_n,\mathbf{y}_n)^2 \end{vmatrix} \quad (1)$$

where $d(\mathbf{x}_i, \mathbf{y}_j) := \|\mathbf{x}_i - \mathbf{y}_j\|_2$ as before.

Definition 3. The Cayley-Menger determinant of points $\{\mathbf{x}_i\}_{i=0}^n$ is defined as:

$$D(\mathbf{x}_0, \dots, \mathbf{x}_n) = D(\mathbf{x}_0, \dots, \mathbf{x}_n; \mathbf{x}_0, \dots, \mathbf{x}_n). \quad (2)$$

The Cayley Menger bi-determinant is related to the signed hyper-volumes V_X and $V_Y \in \mathbb{R}$ of the simplexes formed by points $\{\mathbf{x}_i\}_{i=0}^n$ and $\{\mathbf{y}_i\}_{i=0}^n$, respectively, as follows:

$$D(\mathbf{x}_0, \dots, \mathbf{x}_n; \mathbf{y}_0, \dots, \mathbf{y}_n) = (n!)V_X \cdot (n!)V_Y. \quad (3)$$

The special case where both sets have the same points arranged in the same order was presented in [9] and [16] and is called Cayley-Menger determinant. Equation (3) shows that the Cayley-Menger determinant of the simplex formed by points $\{\mathbf{x}_i\}_{i=0}^n$ is related to its hyper-volume by:

$$D(\mathbf{x}_0, \dots, \mathbf{x}_n) = (n!)^2 V_X^2. \quad (4)$$

We use the previous definitions and relations to obtain a formula for computing barycentric coordinates in n-dimensions from noiseless distance measurements.

[1]An affine frame is a set of points in an affine space such that vectors from one of the points to all others are linearly independent.

Proposition 1. *For each node $i \in \mathcal{V}$, let $\mathcal{N}^i = \{j \in \mathcal{V} | (i,j) \in \mathcal{E}\}$ be the set of neighbor nodes of i. Also, let $\mathfrak{I}^i$ be the index set containing sub-sets of all combinations of $n+1$ neighbors of i which form a complete subgraph of $\mathcal{G}$; i.e. each sub-set k of $\mathfrak{I}^i$, called $\mathfrak{I}_k^i \subset \mathcal{N}^i$, has $n+1$ distinct indexes of neighbors of node i. Then, all possible barycentric coordinates of node i can be arranged row-wise in matrix $L^i \in \mathbb{R}^{|\mathfrak{I}^i| \times m}$ as follows:*

$$[L^i]_{kj} = \qquad\qquad (5)$$
$$\frac{D(\mathbf{x}_{\mathfrak{I}_{k0}^i}, \dots, \mathbf{x}_{\mathfrak{I}_{kn}^i}; \mathbf{x}_{\mathfrak{I}_{k0}^i}, \dots, \mathbf{x}_{\mathfrak{I}_{k,l-1}^i}, \mathbf{x}_i, \mathbf{x}_{\mathfrak{I}_{k,l+1}^i}, \dots, \mathbf{x}_{\mathfrak{I}_{kn}^i})}{D(\mathbf{x}_{\mathfrak{I}_{k0}^i}, \dots, \mathbf{x}_{\mathfrak{I}_{kn}^i})}$$

if $\mathcal{I}_k = j$ or $[L^i]_{kj} = 0$ otherwise.

Proof. The linear system in Definition 1 can be solved for $\boldsymbol{\lambda}$ utilizing Cramer's rule. In this case, each component of L^i is obtained by dividing the signed hyper-volumes of two simplexes:

$$[L^i]_{kj} = \frac{V_{\{\mathbf{x}_{\mathfrak{I}_{k0}^i}, \dots, \mathbf{x}_{\mathfrak{I}_{k,l-1}^i}, \mathbf{x}_i, \mathbf{x}_{\mathfrak{I}_{k,l+1}^i}, \dots, \mathbf{x}_{\mathfrak{I}_{kn}^i}\}}}{V_{\{\mathbf{x}_{\mathfrak{I}_{k0}^i}, \dots, \mathbf{x}_{\mathfrak{I}_{kn}^i}\}}}, \quad (6)$$

if $\mathcal{I}_k = j$ or $[L^i]_{kj} = 0$ otherwise. Pre-multiplying the right side of equation (6) by $\frac{(n!)V_{\{\mathbf{x}_{\mathfrak{I}_{k0}^i}, \dots, \mathbf{x}_{\mathfrak{I}_{kn}^i}\}}(n!)}{(n!)V_{\{\mathbf{x}_{\mathfrak{I}_{k0}^i}, \dots, \mathbf{x}_{\mathfrak{I}_{kn}^i}\}}(n!)}$ does not change its value but allow us to use equations (3) and (4) to arrive at equation (5). $\qquad \square$

Following this method, one can compute all possible barycentric coordinates of node $i \in \mathcal{V}$ with respect to different combinations $n+1$ neighbors, as long as the necessary range measurements are available. Though utilizing all possible combinations of neighbors may be computationally expensive, it allows us to guarantee the localizability of all unknown nodes. Thus, one may solve the relative localization problem by concatenating all matrices L^i, for $i = 1, \dots, m$, forming an overdetermined linear system. We use this approach in our online distributed algorithm described in Sec. V. First, however, we describe the algorithm of Diao *et al.* [11], that defines a generalization of barycentric coordinates using averages. The generalized barycentric coordinates of node i are defined as:

$$\boldsymbol{\lambda}_j^i = \sum_{k=1}^{|\mathfrak{I}^i|} [L^i]_{kj}. \quad (7)$$

Based on equation (7), one can construct the following linear system which relates the generalized barycentric coordinates and the Cartesian coordinates of all nodes. Let $X = [\mathbf{x}_1, \dots, \mathbf{x}_m]^T \in \mathbb{R}^{m \times n}$ and $\Lambda = [\boldsymbol{\lambda}^1, \dots, \boldsymbol{\lambda}^m]^T \in \mathbb{R}^{m \times m}$, i.e., $[\Lambda]_{ij} \triangleq \boldsymbol{\lambda}_j^i$. Then, based on Definition 1, we have $X = \Lambda X$. As the anchor node coordinates $X_a \in \mathbb{R}^{p \times n}$ are known, we can isolate the unknown nodes $X_u \in \mathbb{R}^{q \times n}$ by permuting the rows and columns of the linear system as follows:

$$\begin{bmatrix} X_a \\ X_u \end{bmatrix} = \begin{bmatrix} \Lambda_{aa} & \Lambda_{au} \\ \Lambda_{ua} & \Lambda_{uu} \end{bmatrix} \begin{bmatrix} X_a \\ X_u \end{bmatrix}, \quad (8)$$

where block matrices $\Lambda_{aa} \in \mathbb{R}^{p \times p}$, $\Lambda_{au} \in \mathbb{R}^{p \times q}$, $\Lambda_{ua} \in \mathbb{R}^{q \times p}$ and $\Lambda_{uu} \in \mathbb{R}^{q \times q}$ relate generalized barycentric coordinates among anchor and unknown nodes.

Based on this construction, we can develop a centralized algorithm for relative localization with noiseless measurements by simply solving the linear system in (8) as follows.

Proposition 2. *The centralized noiseless localization problem can be solved by computing the estimate*

$$\tilde{X}_u = \underset{X \in \mathbb{R}^{q \times n}}{\arg \min} ||(\Lambda_{uu} - I)X + \Lambda_{ua}X_a||^2, \qquad (9)$$

which leads to:

$$\tilde{X}_u = -(\Lambda_{uu} - I)^{-1}\Lambda_{ua}X_a. \qquad (10)$$

in the case that $(\Lambda_{uu} - I)$ is invertible.

Diao *et al.* [11] provide conditions for the existence of a unique solution for this problem in 2-D space based on the structure of the network graph. As long as there is a sufficient number of unique paths in the measurement graph from the anchor nodes to every unknown node, the solution to the centralized noiseless relative localization problem in n dimensions would be unique and given by (10). We investigate the noisy measurements case next.

IV. AVERAGING NOISY MEASUREMENTS

Sec. III showed that the entries of the matrix Λ in (8), which specifies the linear system of equations for the unknown nodes, have a nonlinear dependence on the range measurements due to the Cayley-Menger bi-determinant computations. In this section, we consider the relative localization problem in the presence of noisy measurements. The main challenge is that the distribution of the measurement noise $\delta_{ij}(t)$ is, in general, unknown and even if we were to make an assumption on its class (e.g., common choices include Gaussian, Laplace, Rayleigh, or Rice), when it is propagated through the nonlinear barycentric coordinate function in (5), the posterior would not be a standard or stable distribution. We begin by introducing two averaging schemes to mitigate the noise effects and allow Proposition 2 to be applied in the noisy case.

A. Two-phase averaging

We follow the approach of Khan *et al.* [17] to deal with this challenge. Their idea is to leverage the Law of Large Numbers, which ensures the that iid measurements $z_{ij}(t)$, when averaged over time, almost surely converge to the noiseless range measurement $d(\mathbf{x}_i, \mathbf{x}_j)$, and compute the matrix Λ using averaged measurements. In detail, instead of computing $\Lambda(t)$ at each time step t using the raw measurements $\{z_{ij}(t) \mid (i,j) \in \mathcal{E}\}$, we compute $\Lambda(t)$ using the following *averaged range measurements*:

$$\bar{z}_{ij}(t) = \frac{t-1}{t}\bar{z}_{ij}(t-1) + \frac{1}{t}z_{ij}(t), \text{ with } \bar{z}_{ij}(0) = 0. \quad (11)$$

This process, guarantees that after sufficient time passes, $\Lambda(t)$ will be close to the matrix resulting from the true

node barycentric coordinates, and the noisy relative localization problem can again be solved via Proposition 2. Despite the measurement averaging, there may be instances in which large instantaneous noise at time t would lead to an inaccurate $\Lambda(t)$. To mitigate the noise introduced by the most recent measurement, we introduce a second averaging process over the elements of matrix $\Lambda(t)$. In detail, we propose the following recursive steps for computing time-averaged blocks of $\Lambda(t)$:

$$A(t) = [\Lambda_{au}^T(t), (\Lambda_{uu}(t) - I)^T]^T \in \mathbb{R}^{m \times q}$$
$$B(t) = [(\Lambda_{aa}(t) - I)^T, \Lambda_{ua}^T(t)]^T \in \mathbb{R}^{m \times p}$$
$$\bar{A}(t) = \frac{t-1}{t}\bar{A}(t-1) + \frac{1}{t}A(t), \text{ with } \bar{A}(0) = 0 \qquad (12)$$
$$\bar{B}(t) = \frac{t-1}{t}\bar{B}(t-1) + \frac{1}{t}B(t), \text{ with } \bar{B}(0) = 0.$$

Given these definitions, we can return to the approach in Proposition 2 but this time using the time-averaged matrices. In the noisy case, more accurate results can be obtained by using all available equations in (8):

$$\tilde{X}_u = \underset{X \in \mathbb{R}^{q \times n}}{\arg \min} ||\bar{A}(t)X + \bar{B}(t)X_a||^2, \qquad (13)$$

leading to the following least-squares estimate of the unknown node locations:

$$\tilde{X}_u(t) = -\left(\bar{A}(t)^T\bar{A}(t)\right)^{-1}\bar{A}(t)^T\bar{B}(t)X_a. \qquad (14)$$

While this process mitigates the effect of the measurement noise on the computation of the blocks of $\Lambda(t)$ for large enough t, the computation requires centralized information. As a centralized solution to the relative localization problem is not suitable for large networks, in Sec. V we develop a distributed gradient descent algorithm. To motivate the distributed algorithm, we first derive a gradient descent approach for the centralized case with noise measurements.

B. Centralized gradient descent

The gradient of the objective function in (13) is:

$$G(t) := 2\bar{A}(t)^T\left(\bar{A}(t)X + \bar{B}(t)X_a\right). \qquad (15)$$

Thus, we can define an iterative centralized gradient descent algorithm for computing $\tilde{X}_u$ as follows:

$$\tilde{X}_u(t+1) = \tilde{X}_u(t) - \eta(t)G(t), \qquad (16)$$

where $G(t)$ is evaluated at $\tilde{X}_u(t)$ and the step size $\eta(t)$ determines the stability and convergence speed of the algorithm. In general, for fixed step sizes in conjunction with measurement noise, magnitudes of the step size of around 10^{-6} or smaller were needed to guarantee convergence. The algorithm is initialized by computing $\tilde{X}_u(1)$ from the first set of range measurements via (14). One choice for $\eta(t)$ that we found to work particularly well is based on the Barzilai-Borwein method [23]:

$$\eta(t) := \frac{\mathbf{tr}\left((\tilde{X}_u(t) - \tilde{X}_u(t-1))^T(G(t) - G(t-1))\right)}{\mathbf{tr}\left((G(t) - G(t-1))^T(G(t) - G(t-1))\right)}.$$

As can be seen in Fig 3, this dynamic step size behaves similarly to a well adjusted fixed step size, but it circumvents the need to manually search for optimal step size values.

V. Distributed Online Gradient Descent

The centralized algorithms presented in Sec. III rely on the generalized barycentric coordinates (7) proposed by Diao *et al.* [11]. Instead of summing the rows of the matrices $L^i \in \mathbb{R}^{|\mathfrak{I}^i| \times m}$ of the standard barycentric coordinates as in (7), we can directly define an overdetermined linear system in place of (8). In detail, the barycentric coordinates $L^i \in \mathbb{R}^{|\mathfrak{I}^i| \times m}$ for every node $i \in \mathcal{V}$ satisfy:

$$L^i X = \mathbf{1}_{|\mathfrak{I}^i|} \mathbf{x}_i^T \quad \Leftrightarrow \quad \left(L^i - \mathbf{1}_{|\mathfrak{I}^i|} \mathbf{e}_i^T \right) X = 0 \qquad (17)$$

where $X = \begin{bmatrix} X_a^T & X_u^T \end{bmatrix}^T \in \mathbb{R}^{m \times n}$ as before, $\mathbf{1}$ is a vector, whose elements are all equal to 1, and $\mathbf{e}_i \in \mathbb{R}^n$ is a standard basis vector. Defining $\mathbb{L}^i = \left(L^i - \mathbf{1}_{|\mathfrak{I}^i|} \mathbf{e}_i^T \right) \in \mathbb{R}^{|\mathfrak{I}^i| \times m}$ and decomposing it into block matrices associated with the anchors and unknown nodes as before, allows us to write the above constraints in the noiseless case as:

$$\mathbf{0}_{|\mathfrak{I}^i| \times n} = \begin{bmatrix} \mathbb{L}_{aa}^i & \mathbb{L}_{au}^i \\ \mathbb{L}_{ua}^i & \mathbb{L}_{uu}^i \end{bmatrix} \begin{bmatrix} X_a \\ X_u \end{bmatrix}, \quad i = 1, \ldots, m. \qquad (18)$$

Let $\mathbb{A}^i = \begin{bmatrix} \mathbb{L}_{au}^i \\ \mathbb{L}_{uu}^i \end{bmatrix}$ and $\mathbb{B}^i = \begin{bmatrix} \mathbb{L}_{aa}^i \\ \mathbb{L}_{ua}^i \end{bmatrix}$ be the columns of $\mathbb{L}^i$ corresponding to X_a and X_u, respectively. In the presence of noisy measurements, the constraints in (18) can only be satisfied approximately. Hence, we use the same two-phase averaging process as in eq. (12) in Sec. IV-A to define $\bar{\mathbb{A}}^i(t)$ and $\bar{\mathbb{B}}^i$ and aim to compute a least-squares estimated for X_u:

$$\tilde{X}_u(t) := \underset{X \in \mathbb{R}^{q \times n}}{\arg\min} \sum_{i=1}^m \left\| \bar{\mathbb{A}}^i(t) X + \bar{\mathbb{B}}^i(t) X_a \right\|^2 \qquad (19)$$

Notice that each matrix $\mathbb{L}^i$ can be computed locally in each node i with 2-hop information. But each neighbor already stores information about its neighbors. Therefore, all necessary computation can be done with only 1-hop communication. The structure of this interaction is defined with respect to a communication matrix W, selected based on Xiao *et al.* [24] to be

$$W = I - L_{\mathcal{G}} / \max\{eig(L_{\mathcal{G}})\}, \qquad (20)$$

where $L_{\mathcal{G}}$ is the Laplacian matrix of graph $\mathcal{G}$ and $eig(\cdot)$ computes the eigenvalues of the respective matrix.

Proposition 3. *Given anchor node locations $\mathcal{X}_a \subset \mathcal{X}$ of a static sensor network and noisy range measurements $z_{ij}(t) = d(\mathbf{x}_i, \mathbf{x}_j) + \delta_{ij}(t) \in \mathbb{R}, \ \forall (i,j) \in \mathcal{E} \ and \ t = 0, \ldots, T$. We can estimate the locations of all unknown nodes $\mathcal{X}_u \subset \mathcal{X}$ at each node i with step sizes $\eta(t)$ by performing local state updates*

$$X_u^i(t+1) = \sum_{j=1}^m [W]_{ij} X_u^j(t) - \eta(t) G^i(t), \qquad (21)$$

where the gradient evaluated at $X_u^i(t)$ is

$$G^i(t) = 2\bar{\mathbb{A}}^i(t)^T \left(\bar{\mathbb{A}}^i(t) X_u^i(t) + \bar{\mathbb{B}}^i(t) X_a \right) \qquad (22)$$

and W is defined in (20).

Proof. The online mirror descent technique in [21] is developed to track the minimizer of a global objective function (at each time t), where the global function can be written as a sum of m local functions. It is evident that the problem formulation given in equation (19) conforms to the setup proposed in [21]. Moreover, all the necessary assumptions of the method are satisfied when using the Euclidean norm as the Bregman Divergence.

Consequently, according to this method, the online distributed algorithm becomes

$$x_{u_i}(t+1) = \arg\min \left\{ \eta_t \langle x, \nabla f_{i,t}(x_{u_i}(t)) \rangle + \frac{1}{2} \|x - y_i(t)\|_2^2 \right\}$$

$$y_i(t) = \sum_{j=1}^m [W]_{ij} x_{u_j}(t+1),$$

where x_{u_i} and $\nabla f_{i,t}$ are derived by vectorizing X_u^i and G^i in (21) and (22), respectively. Solving the above leads to (21), thereby concluding the proof. $\square$

The previous proposed solution of using Barzilai-Borwein method [23] does not work in a distributed environment. We are using a small fixed step size, 10^{-6}, in all distributed cases.

Remark 1 (Communication cost). Suppose that data is kept in floats, usually 4 bytes. Then, every iteration each node i would receive and transmit $un|\mathcal{N}^i| + (n+1)|\mathcal{I}^i|$ floats.

VI. Simulations results

We carried out simulations comparing the performance of the centralized two-phase averaging algorithm (Sec. IV-A), the centralized gradient descent algorithm (Sec. IV-B) and the distributed online gradient descent (Sec. V). Fig. 2 shows a randomly generated static network with 4 anchors and 10 unknown nodes, whose ground truth position lie on a cube with side length of 10 units. The maximum measurement range of each node was $r_i = 8$ units and random Gaussian noise with variance equal to either a tenth of the actual distance between nodes or the averaged valued of these proportional variances were applied to the range measurements. Proportional variances characterize that larger distances have larger noise, while a constant one says that the sensors maintain the same amount of noise throughout its range. In this simulated case, our results had no significant discrepancies depending on these choices. The node coordinate estimates computed by the first two algorithms over time are also shown, as well as the estimates locally computed by the third algorithm in a randomly selected node.

There are two fundamental differences between the centralized and distributed algorithms. First, while the centralized algorithm has a single set of estimates for the unknown node locations, the distributed algorithm maintains a different set of locations estimates for each node in the network. Second, the amount of information available to each node in the distributed algorithm is different compared to the centralized case and depends on the network communication structure.
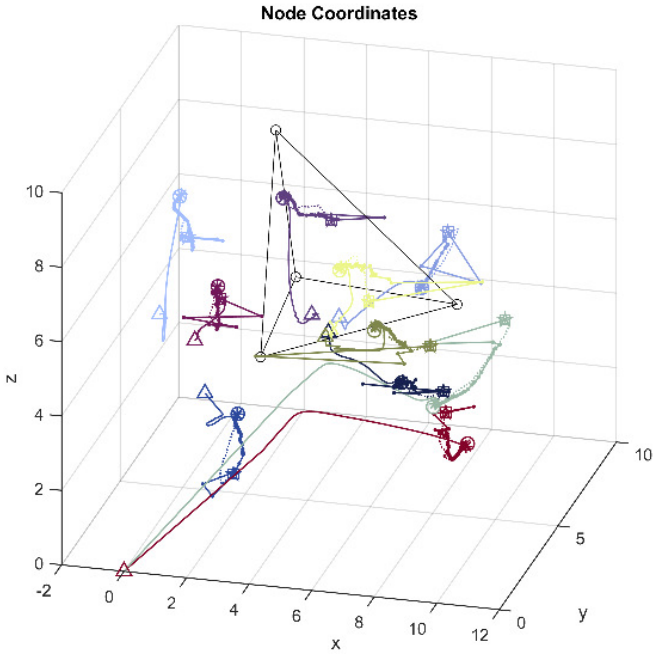
Fig. 2. Randomly generated sensor network with 10 unknown nodes and 4 anchors and trajectories of the node coordinate estimates over time. Each color represents a different unknown node. The dotted, dot dashed and full lines beginning with $\{\square, \hexagon, \triangle\}$ and ending with $\{\diamond, \times, \divideontimes\}$ show the estimates over time obtained by the two-phase averaging (Sec. IV-A), centralized gradient descent (Sec. IV-B) and distributed online gradient descent (Sec. V) algorithms, respectively. The ground truth node locations are marked by circles, while the black tetrahedron shows the convex hull of the anchor nodes. As each node in the distributed algorithm computes its own unknown node location estimates, it is not possible to show the trajectories of the estimates of all nodes. Instead, we show the trajectories of the estimated node locations computed by one randomly chosen node.

These facts are clearly seen in the results. For example, note that the two-phase averaging and the centralized gradient descent algorithms provide estimate trajectories that remain close to each other, while some estimates obtained by the distributed gradient descent algorithm follow very different trajectories. Moreover, it is important to note that while the first two algorithms have the same initial estimate, the same does not occur for the distributed algorithm, despite computing initial estimates in similar ways. This behavior is due to the fact that the distributed algorithm uses only the first noisy range measurements to initialize its estimates. Both centralized algorithms are able to provide better initial guesses for all node locations, while the distributed algorithm provides poorer results for nodes with fewer neighbors or at a further distance from the anchors.

Fig. 3 shows the Root Mean Square Error (RMSE) of the node coordinate estimates over time for the three algorithms. Both centralized algorithms arrive at satisfactory location estimates with less than 5000 iterations, while the same can not be said for the distributed gradient descent algorithm. It is clear that the initial estimates in the distributed case are worse than the ones in the centralized algorithms, which contributes to its slower convergence. Another important factor that impacts our distributed gradient descent method is the
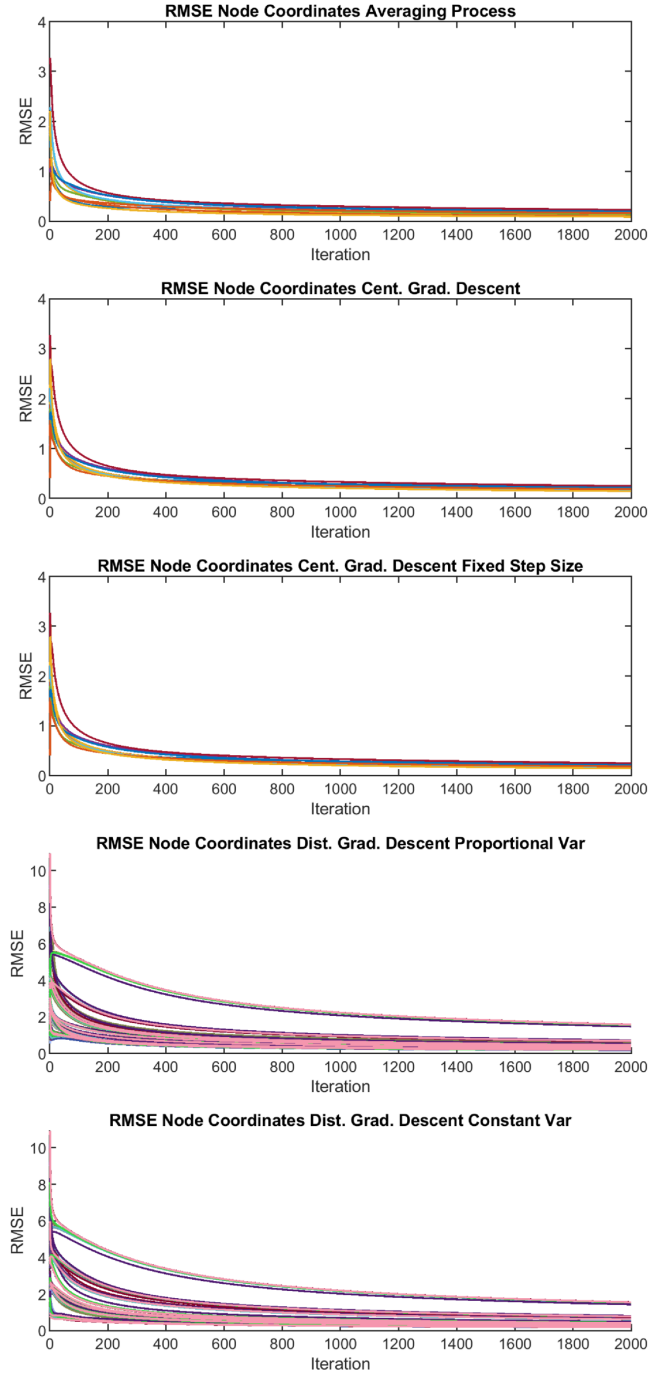

Fig. 3. Root Mean Square Error (RMSE) of the node coordinate estimates over time for the two-phase averaging (Sec. IV-A), the centralized gradient descent with Barzilai-Borwein and fixed step sizes (Sec. IV-B) and distributed online gradient descent with proportional and constant variances (Sec. V) algorithms. As each node in the distributed algorithm computes its own unknown node location estimates, we present RMSE values for the estimates of each node on the same plot.

choice of the learning rate, $\eta(t)$. A small learning rate will provide a more stable but slower convergence. So far, while we used an $\eta(t)$ based on the Barzilai-Borwein method [23] for the centralized gradient descent algorithm with good results, we were not able to compute $\eta(t)$ online in a way that

provides faster convergence rates while guaranteeing stability of the process for different networks. Therefore, we used a small fixed η only for simulations involving our distributed online gradient descent algorithm.

VII. Conclusion

This paper presented a closed-form expression for computing barycentric coordinates in arbitrary n-dimensional node configurations based on Cayley-Menger bi-determinants. This enabled a formulation of the relative localization problem as a linear system defined in terms of barycentric coordinates. Based on this construction, we developed centralized and distributed algorithms capable of handling noise-free or noisy measurements with arbitrarily distribution in n dimensional sensor networks with arbitrary anchor configurations. Future work will focus on trade-offs between computational efficiency and localization accuracy by analyzing the effect of the number of neighbor subsets of size $n+1$ used for barycentric coordinate computations. Extensions to mobile networks and mixed bearing-only and range-only measurements across the network are of interest as well.

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